

A Physical Approach to Polya's Conjecture

Jingbo Wang

Department of Applied Physics, Xi'an Jiaotong University, Xi'an, 710049,

People's Republic of China

`shuijing31@gmail.com`

ABSTRACT: The similarity between the Polya's conjecture and the Bonomol'nyi bound remind us to consider a physical approach to Polya's conjecture. We conjecture a duality between the waves and the soliton solutions on the surface. We consider the special case in the disc.

KEYWORDS: Polya's conjecture, wave-soliton duality, Bonomol'nyi bound.

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1. Introduction

Let $\Omega \subset R^2$ be a domain with a boundary $\partial\Omega$. Define the Dirichlet Laplacian operator Δ . Then the eigenvalue problem of the Laplacian on the Ω is

$$\begin{cases} \Delta\Psi_n + \lambda_n\Psi_n = 0 \\ \Psi_n|(\partial\Omega) = 0 \end{cases} \quad (1.1)$$

Then Polya's conjecture[1] says that: for every λ_n , it obey the inequality:

$$\lambda_n \geq \frac{4\pi}{A}n. \quad (1.2)$$

Where A is the area of the domain. Polya himself proved that this conjecture is right for the tiling domain, but the general case for the arbitrary domain is still open. Up to now, the best result is obtained by Li and Yau[2]:

$$\sum_{j=1}^n \lambda_j \geq \frac{2\pi n^2}{A}, n \in \mathbb{N} \quad (1.3)$$

The main goal of this paper is to suggesting a physical approach to Polya's conjecture, based on the similarity between the inequality 1.2 and the Bonomol'nyi bound in soliton theory[3].

2. Nonlinear sigma model

In this section, we briefly review the nonlinear $O(3)$ σ model in $(2+1)$ dimension and its soliton solution[4]. It involve three real scalar fields $\phi(x^\mu) \equiv \{\phi_a(x^\mu), a = 1, 2, 3, x^\mu = (t, x, y)\}$ with the constrain:

$$\phi_a \phi_a = 1. \quad (2.1)$$

Thus, the fields lie on the unit sphere S_2 .

Subject to this constrain the lagrangian density reads

$$\mathcal{L} = \frac{1}{4} \partial_\mu \phi_a \partial^\mu \phi_a. \quad (2.2)$$

which is invariant under global $O(3)$ rotations in internal space. From the lagrangian we can get the equation of motion

$$\partial^\mu \partial_\mu \phi_a - (\phi_b \partial^\mu \partial_\mu \phi_b) \phi_a = 0. \quad (2.3)$$

which for the static case reduces to

$$\Delta \phi_a - (\phi_b \Delta \phi_b) \phi_a = 0. \quad (2.4)$$

From the constrain equation 2.1, we can get

$$\phi_b \Delta \phi_b = -\nabla \phi_b \bullet \nabla \phi_b. \quad (2.5)$$

so the equation 2.4 changes to

$$\Delta \phi_a + (\nabla \phi_b \bullet \nabla \phi_b) \phi_a = 0. \quad (2.6)$$

Define the energy density $\mathcal{E}(x, y) = \nabla \phi_b \bullet \nabla \phi_b$, then the potential energy is given by

$$V = \frac{1}{4} \int (\nabla \phi_b \bullet \nabla \phi_b) dx dy = \frac{1}{4} \int \mathcal{E}(x, y) dx dy. \quad (2.7)$$

Notice the similarity between the equation 1.1 and 2.6.

The problem is completely specified by giving the boundary condition. We take

$$\lim_{r \rightarrow \infty} \vec{\phi}(r, \theta) = \vec{\phi}^0. \quad (2.8)$$

where the unit vector $\vec{\phi}^0$ is a constant vector. We will take it to be $\vec{\phi}^0 = (0, 0, 1)$.

For a soliton solution, we have the Bonomol'nyi bound for the energy of the soliton,

$$E = V \geq 2\pi |Q| \quad (2.9)$$

where the Q is called the topological charge, or winding number:

$$Q = \frac{1}{4\pi} \int_{\Omega} \vec{\phi} \bullet (\partial_x \vec{\phi} \times \partial_y \vec{\phi}) dx dy \in \mathbb{N}. \quad (2.10)$$

The mean value of the energy density of the soliton with winding number $Q = n$ will be

$$\bar{\mathcal{E}} = \frac{4V}{A} \geq \frac{8\pi}{A} |Q| = \frac{8\pi}{A} n \quad (2.11)$$

If we can find some function between the eigenfunction Ψ_n for the domain and the n -soliton solution $\vec{\phi}^n$ of the sigma model on the domain,

$$\Psi_n = F(\vec{\phi}^n), \quad (2.12)$$

that is the duality between the waves on the domain and the solitons on the domain, maybe we can find the proof of the Polya's conjecture.

3. The Polya's conjecture on the disc

In this section, we consider the special case of the Polya's conjecture, the conjecture on the disc with the radius R_0 .

The eigenvalue problem of the Laplacian operator can be solved explicitly. The eigenfunction can be written as

$$\Psi_{mn}(r, \theta) = c J_m(\lambda_{mn} r) (a \cos m\theta + b \sin m\theta). \quad (3.1)$$

where $J_m(x)$ is the Bessel function, and the eigenvalues

$$\lambda_{mn}^2 = (\alpha_m^n / R_0)^2 \quad (3.2)$$

where α_m^n is the n -zero of the $J_m(x)$. The radial function $R(r)$ satisfy the Bessel function:

$$R'' + \frac{R'}{r} + (\lambda_m^2 - \frac{m^2}{r^2}) R = 0. \quad (3.3)$$

This equation also can be obtained by the energy functional

$$E = \int_0^{R_0} (R'^2 + \frac{m^2 R^2}{r^2} - \lambda_m^2 R^2) r dr \quad (3.4)$$

through the Euler-Lagrange equation.

For a special eigenfunction $J_m(\lambda_{mn}r)$, the energy functional is given by

$$E = \int_0^{R_0} (\lambda_{mn}^2 (J'_m(\lambda_{mn}r))^2 + (\frac{m^2}{r^2} - \lambda_{mn}^2) J_m^2(\lambda_{mn}r)) r dr = 1/2 \int_0^{\lambda_{mn}R_0} (J_{m-1}^2(x) + J_{m+1}^2(x) - 2J_m^2(x)) x dx \quad (3.5)$$

Next we consider the nonlinear sigma model on the disc. With the stereographic projection of the S_2 , we can get

$$\vec{\phi} = \frac{1}{1 + uu^*} (u + u^*, -i(u - u^*), uu^* - 1) \quad (3.6)$$

and the complex function $u = \frac{\phi_1 + i\phi_2}{1 - \phi_3}$.

With the hedgehog ansatz

$$\vec{\phi}(r, \theta) = (\sin f(r) \cos n\theta, \sin f(r) \sin n\theta, \cos f(r)),$$

where the function $f(r)$ is called the profile function and satisfy some boundary condition, which is

$$f(0) = \pi,$$

and

$$\lim_{r \rightarrow R_0} f(r) = 0.$$

The n is the winding number. we get

$$u(r, \theta) = \tan(f(r)/2) \exp(in\theta)$$

and the equation of motion for the complex field is

$$\Delta u - \frac{2\nabla u \bullet \nabla u}{1 + uu^*} u^* = 0. \quad (3.7)$$

So the equation for the $f(r)$ is

$$f'' + \frac{f'}{r} - \frac{n^2 \sin 2f}{2r^2} = 0. \quad (3.8)$$

Obviously this equation is a nonlinear differential equation and can be solved numerically (with the shooting method). If the value of $f(r)$ is obtained then we can use it to calculate the static energy and so on. As before, this equation can be obtained by the energy functional

$$E = 2\pi \int_0^{R_0} (f'^2 + \frac{n^2 \sin^2 f}{r^2}) r dr. \quad (3.9)$$

Notice the similarity and the difference between the two energy functional 3.4 and 3.9.

4. The duality

We want to build the duality between the eigenfunctions and the soliton solutions on the disc, and from this duality we relate the eigenvalues of the Laplacian operator to the energy of the dual soliton, and then from the Bonomol'nyi bound we will get the Polya's conjecture.

In the disc case, we assume the dual soliton correspondence to the eigenfunction Ψ_{mn} is the N -soliton $\vec{\phi}_N$, then the natural question is how to relate those two functions and get the function $N(m, n)$. Since we have no explicit form of the soliton solution, we must guess. Finally we want to build the relation between the energy of those two solutions, such as

$$E_{wave} \geq E_{soliton} \geq 2\pi N. \quad (4.1)$$

and we need much more work to achieve this goal.

5. Conclusion

In this paper, we just give a general framework for solving the Polya's conjecture. We hope that our method can be applied to other related problems in the geometrical analysis, such as: the first eigenvalue obey the inequality

$$\lambda_1 \geq \frac{a}{r_\Omega^2},$$

and the gap between the first two nonzero eigenvalues. We notice that the eigenvalues have the dimension of the energy density.

On the physical side, some people believe that the fundamental particles in our universe should be described by the soliton solution of some nonlinear equation[5], such as the Skyrme model for the hadrons in QCD[6]. The duality between the wave and the soliton can shine some light on those ideas.

For the Neumann boundary, the the Polya's conjecture has the form of

$$\mu_n \leq \frac{4\pi}{A}n.$$

We didn't find any similar physical results. On the other hand, if we can find the dual Dirichlet system such that they obey

$$\lambda_n + \mu_n = \frac{8\pi}{A}n,$$

then from the Polya's conjecture on the Dirichlet boundary we can get the problem for the Neumann boundary. The T-duality in string change the Dirichlet boundary to the Neumann boundary, so we guess this duality should play some role in the Neumann boundary problem.

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